SOME COMPLEMENTS TO BROUWER'S FIXED POINT THEOREM

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ABSTRACT

The sets which can be the fixed points of a continuous function or a homeomorphism of B^n are investigated.

We present some elementary complements to Brouwer's fixed point theorem in n-space.

Let $B^n =$ all points $P = (x_1, \cdots, x_n)$ with $\| P \|^2 = \sum_1^n x_i^2 \leqq 1$ and $S^{n-1} =$ all P with $\| P \| = 1$. If $f\colon B^n \to B^n$ is any continuous map of B^n into itself, the *fixed point set* A of f is the set of all $P \in B^n$ such that $f(P) = P$. Clearly, A is closed, and by Brouwer's theorem, non-empty.

THEOREM 1. *For any $n \geqq 1$ and any non-empty closed set $A \subset B^n$, there is a continuous map $f\colon B^n \to B^n$ with A as its fixed point set.*

Proof. Define

$$d(P, A) = \inf_{Q \in A} \| P - Q \|.$$

Then $d(P,A)$ is a continuous function of P, and $d(P,A) = 0$ iff $P \in A$.

Choose any $Q \in A$, and define $f\colon B^n \to B^n$ by setting

$$f(P) = \begin{cases} P + d(P,A) \dfrac{(Q-P)}{\| P - Q \|} & \text{for } P \neq Q, \\ Q & \text{for } P = Q. \end{cases} \tag{1}$$

Then f is continuous and has A as its fixed point set.

THEOREM 2. *For any odd n there is a non-empty closed set $A \subset B^n$ which is not the fixed point set of any homeomorphism $f\colon B^n \to B^n$.*

Proof. Let A consist of all points P with $\| P \| \leqq 1/2$. Suppose $f\colon B^n \to B^n$ is a homeomorphism with A as its fixed point set. Consider the family of continuous maps $f_t\ S^{n-1} \to S^{n-1}$ defined by setting

Received April 15, 1967.

$$f_t(P) = \frac{f(tP)}{\|f(tP)\|} \qquad \left(\frac{1}{2} \leqq t \leqq 1\right);$$

then

$$f_{1/2}(P) = P, \qquad f_1(P) = f(P).$$

Hence the restriction of f to S^{n-1} is homotopic to the identity yet has no fixed points, which is impossible for n odd.

THEOREM 3. *For any non-empty closed set $A \subset B^2$, there exists a homeomorphism $f: B^2 \to B^2$ with A as its fixed point set.*

Proof. *Case* 1. A contains an interior point of B^2, which we may assume to be the origin. Define $f: B^2 \to B^2$ by setting for any $P = (x_1, x_2) \in B^2$, $f(P) = (x_1', x_2')$ with

$$\begin{aligned} x_1' &= x_1 \cos t + x_2 \sin t \\ x_2' &= -x_1 \sin t + x_2 \cos t \end{aligned} \qquad \text{where } t = d(P, A). \tag{2}$$

Clearly, f is continuous, with A as its fixed point set, and it is easy to verify that f is a homeomorphism of B^2.

Case 2. A contains a boundary point of B^2, which we may assume to be the point $(1,0)$. We then replace (2) by

$$\begin{aligned} x_1' - r &= (x_1 - r)\cos t + x_2 \sin t, \quad &\text{where} \quad r^2 = x_1{}^2 + x_2{}^2 \\ x_2' &= -(x_1 - r)\sin t + x_2 \cos t, \quad &t = d(P, A), \end{aligned} \tag{3}$$

and argue as before.

REMARKS. 1. Theorem 3 is true for any B^{2n}, at least in Case 1. To see this, define $f: B^{2n} \to B^{2n}$ by putting $f(P) = (x_1', x_2', \cdots, x_{2n}')$ where x_1', x_2' are defined by (2), x_3', x_4' by (2) with 1 replaced by 3 and 2 replaced by 4, etc. The construction of f in Case 2 for arbitrary B^{2n} remains to be supplied.

2. By taking as B^3 the points P with $x_1{}^2 + x_2{}^2 + (x_3 - 1)^2 \leqq 1$, considering the sections of this by planes through the x_1-axis, and applying the transformation analogous to (3) to each of these, we see that if A is a closed subset of B^3 containing at least one boundary point of B^3 (in this case, the origin), then there exists a homeomorphism $f: B^3 \to B^3$ with A as its fixed point set. Presumably the same holds for any odd n (certainly for $n = 1$).

I am indebted to A. Dvoretzky and P. A. Smith for helpful suggestions.

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